

Symmetries, Quotients and Kähler-Einstein metrics

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Abstract

We consider Fano manifolds M that admit a collection of finite automorphism groups G_1, \dots, G_k , such that the quotients M/G_i are smooth Fano manifolds possessing a Kähler-Einstein metric. Under some numerical and smoothness assumptions on the ramification divisors, we prove that M admits a Kähler-Einstein metric too.

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1 Introduction

The aim of this paper is to provide new examples of Kähler-Einstein metrics of positive scalar curvature. The existence of such a metric on a Fano manifold is a subtle problem, due to the presence of obstructions, that have been discovered during the years, beginning with Matsushima's theorem in 1957, Futaki invariants in 1982, Tian's theorem stating that Kähler-Einstein manifolds of positive scalar curvature are semistable (see [24, Theorem 8.1]), up to Donaldson's result

[11, Corollary 4], which shows that the existence of Kähler-Einstein metrics (even more generally of a Kähler constant scalar curvature metric) forces the algebraic underlying manifolds to be asymptotically stable (see also [1]).

Existence theorems on the other hand are always very hard. The only necessary and sufficient condition, established by Tian, is of a truly analytic character. It says that a Fano manifold M admits a Kähler-Einstein metric, if and only if an integral functional F defined on Kähler metrics in the class $c_1(M)$ is proper (see Theorem 2.1 below). The equivalence of properness of F with the algebraic stability of the underlying manifold, in an appropriate sense, would represent the final solution of the problem, but is still unknown. (This has been suggested by Yau, and made precise by Tian, who has also proved that properness implies stability.) Work in progress by Paul and Tian [18] indicates a new stability condition as a candidate for the equivalence with the existence of a Kähler-Einstein metric.

Although by now there is a good deal of examples, the only broad class of manifolds for which the problem is solved is the one of toric Fano manifolds, thanks to a recent theorem of Xujia Wang and Xiaohua Zhu ([27], see also Donaldson's work [12] for related results for extremal metrics). Otherwise, even for manifolds that are deceptively simple from the algebro-geometric point of view, one has often no clue on how to check the properness of F , and finding the metric. The case of Del Pezzo surfaces is quite eloquent from this point of view, as the reader of [23] might verify. Another striking example of the difficulties on which one suddenly runs, is the hypersurface case. Indeed, it is expected that any smooth Fano hypersurface has a Kähler-Einstein metric, nonetheless the only ones for which this is known are the ones lying in a suitable small analytic neighbourhood of the Fermat's hypersurfaces (see [25, p. 85-87]). In fact a standard implicit function theorem argument shows that the Kähler-Einstein condition is open in the moduli space in the analytic topology, provided the automorphism group is finite. This remark can be applied also to some of the examples discussed below.

In trying to construct explicit examples a good help comes from having many holomorphic symmetries to work with. This has been crucial for example in estimating the so called α -invariant for some Del Pezzo surfaces with reductive automorphisms group. This has been the heart of the work of Tian-Yau [26, Proposition 2.2].

The aim of this paper is to use in a different way the symmetries

of the manifold to prove existence of Kähler-Einstein metrics, inspired by Tian's work on Fermat hypersurfaces. In Section 2 we study the behaviour of properness of F_ω (see p. 2) in presence of a Galois covering and find conditions under which the existence of a Kähler-Einstein metric on the base allows one to prove a version of properness, and thus existence, on the covering space. We find algebraic conditions on the covering maps (Theorems 2.3 and 2.6) ensuring that the desired inequalities hold on the covering space. In 3 we show how this can be used to prove the existence of Kähler-Einstein metrics on some classes of Fano manifolds, chosen from the lists of Del Pezzo manifolds, and Fano threefolds with $\text{Pic} = \mathbb{Z}$ (see [13, p. 214-215]). Our examples include:

- a) hypersurfaces of the form $\{x_0^d + \dots + x_{k-1}^d + f(x_k, \dots, x_{n+1}) = 0\} \subset \mathbb{P}^{n+1}$ where f is a homogeneous polynomial of degree d , and $k > n + 2 - d$;
- b) n -dimensional intersections of hypersurfaces of the same form as above, all of the same degree d and with $k > n + 2 - d$;
- c) arbitrary intersections of two (hyper)quadrics;
- d) double covers of \mathbb{P}^n ramified along a smooth hypersurface of degree $2d$ with $\frac{n+1}{2} < d \leq n$;
- e) double covers of the n -dimensional quadric $Q_n \subset \mathbb{P}^{n+1}$ with smooth branching locus cut out by a hypersurface of degree $2d$ with $\frac{n}{2} < d < n$.

(See section 3.) Example (a) generalises Tian's theorem about Fermat's hypersurfaces. Examples (a), (b), (d) and (e) give positive-dimensional *algebraic families* of Kähler-Einstein manifolds. This becomes even more striking in example (c) since every element in the moduli of such manifolds has a Kähler-Einstein metric. A particular case of (c) (the intersection of two specific quadrics in \mathbb{P}^5) had been previously studied by Alan Nadel (see [17, p. 589]).

Some interesting questions arise naturally from these results. In the first place, when a finite group G acts on an algebraic manifold M , the quotient M/G can always be endowed with the structure of a complex analytic orbifold. We believe that our theorems can be generalised to cover this case, provided the quotient admits a Kähler-Einstein *orbifold* metric. Nevertheless there are few examples of Kähler-Einstein orbifolds (see e.g. [9], [14], [4]), and it is probably hard to apply our results to coverings with orbifold base.

From a different perspective, in light of our results (c)-(f), one could study the Weil-Petersson geometry of the moduli spaces of these new families, or one can try to generalise Mabuchi and Mukai's results ([16]) on compactification of moduli spaces. A situation which seems geometrically appealing is the one of the intersection of two quadrics (which is in fact Mabuchi-Mukai's case in dimension 2). A classical result says the moduli space of the intersection of two quadrics in \mathbb{P}^{2n+3} is isomorphic to the moduli space of hyperelliptic curves of genus n (see [2] and reference therein). Therefore this moduli space inherits two Weil-Petersson geometries, one coming from the Kähler-Einstein metrics on the intersection of quadrics, the other from Poincaré metrics on curves. It would be interesting to compare them.

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2 Existence theorems on covering spaces

Let M be a compact n -dimensional Kähler manifold and ω a smooth closed $(1,1)$ -form on M such that

$$\langle [\omega]^n, [M] \rangle = \int_M \omega^n > 0.$$

When $\varphi \in C^\infty(M)$ put $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi$. Define the following functionals on $C^\infty(M)$:

$$I_\omega(\varphi) = \frac{1}{\langle [\omega]^n, [M] \rangle} \int \varphi(\omega^n - \omega_\varphi^n) \quad (1)$$

$$J_\omega(\varphi) = \int_0^1 \frac{I_\omega(s\varphi)}{s} ds \quad (2)$$

$$F_\omega^0(\varphi) = J_\omega(\varphi) - \frac{1}{\langle [\omega]^n, [M] \rangle} \int \varphi \omega^n. \quad (3)$$

When no confusion is possible, we will write $V = \langle [\omega]^n, [M] \rangle$. For the reader's convenience we recall the following equivalent definitions of these functionals.

Lemma 2.1 *If M and ω are as above, and $\varphi \in C^\infty(M)$, then*

$$J_\omega(\varphi) = -\frac{n!}{\langle [\omega]^n, [M] \rangle} \sum_{p=1}^n \frac{1}{(n-p)!(p+1)!} \int_M \varphi \omega^{n-p} (\mathbf{i} \partial \bar{\partial} \varphi)^p = \quad (4)$$

$$= -\frac{1}{\langle [\omega]^n, [M] \rangle} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_M \varphi \mathbf{i} \partial \bar{\partial} \varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1} = \quad (5)$$

$$= \frac{1}{\langle [\omega]^n, [M] \rangle} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_M \mathbf{i} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1} \quad (6)$$

and

$$F_\omega^0(\varphi) = -\frac{n!}{\langle [\omega]^n, [M] \rangle} \sum_{p=0}^n \frac{1}{(n-p)!(p+1)!} \int_M \varphi \omega^{n-p} (\mathbf{i} \partial \bar{\partial} \varphi)^p. \quad (7)$$

Proof (sketch). To prove (4) expand $\omega_{s\varphi}^n = (\omega + s \mathbf{i} \partial \bar{\partial} \varphi)^n$ in powers of s and use the result to compute $I_\omega(s\varphi)$ in (2). As for (5) compute $I_\omega(s\varphi)$ in (2) using the fact that

$$\omega^n - \omega_{s\varphi}^n = (\omega - \omega_{s\varphi}) \sum_{q=0}^{n-1} \omega^{n-q-1} \wedge \omega_{s\varphi}^q.$$

Substituting $\omega_{s\varphi} = s\omega_\varphi + (1-s)\omega$ and expanding $\omega_{s\varphi}^q$ yields

$$J_\omega(\varphi) = -\frac{1}{\langle [\omega]^n, [M] \rangle} \sum_{p=0}^{n-1} C_p \int_M \varphi \mathbf{i} \partial \bar{\partial} \varphi \wedge \omega^{n-1-p} \wedge \omega_\varphi^p$$

where

$$C_p = \sum_{q=p}^{n-1} \binom{q}{p} \int_0^1 s^{p+1} (1-s)^{q-p} ds.$$

This can be computed using the combinatorial identities

$$\begin{aligned} \int_0^1 s^{p+1} (1-s)^k ds &= \frac{(p+1)!k!}{(p+k+2)!} \\ \sum_{k=0}^{n-p-1} \frac{p+1}{(p+k+1)(p+k+2)} &= \frac{n-p}{n+1} \end{aligned}$$

and gives the desired result. To get (6) it is enough to integrate by parts, using that ω is closed and M is Kähler. Finally (7) is an immediate consequence of (4). Q.D.E.

Formula (7) says that F^0 coincides (up to a constant factor) with the functional called I by other authors. Compare with eq. (25) in [10] where Donaldson gives a nice geometric interpretation of F^0 .

Lemma 2.2 *Let M and ω be as above. If λ is a positive constant then*

$$F_{\lambda\omega}^0(\lambda\varphi) = \lambda F_{\omega}^0(\varphi). \quad (8)$$

Let ω_0 be a closed (1,1)-form such that $\langle [\omega_0], [M] \rangle > 0$. Given $\varphi_{01}, \varphi_{12} \in C^\infty(M)$ put $\omega_1 = \omega_0 + i\partial\bar{\partial}\varphi_{01}$, $\varphi_{02} = \varphi_{01} + \varphi_{12}$. Then

$$F_{\omega_0}^0(\varphi_{02}) = F_{\omega_0}^0(\varphi_{01}) + F_{\omega_1}^0(\varphi_{12}). \quad (9)$$

For the proof see [25, pp. 60f].

Assume from now on that M is a Fano manifold and ω is a Kähler metric in the class $2\pi c_1(M)$. Then $V = \langle [\omega]^n, [M] \rangle = n! \text{vol}(M)$. Let $f = f(\omega)$ be the unique function on M satisfying

$$\text{Ric}(\omega) = \omega + i\partial\bar{\partial}f(\omega), \quad \int_M e^{f(\omega)} \omega^n = V. \quad (10)$$

Define $A_\omega, F_\omega : C^\infty(M) \rightarrow \mathbb{R}$ by

$$A_\omega(\varphi) = \log \left[\frac{1}{V} \int_M e^{f(\omega) - \varphi} \omega^n \right]$$

$$F_\omega(\varphi) = F_\omega^0(\varphi) - A_\omega(\varphi).$$

Although these functionals (as well as the ones defined before) are defined on the whole of $C^\infty(M)$, their interest for Kähler-Einstein metrics lies in their behaviour on a smaller space, whose definition we now recall.

Let G be a compact group of isometries of (M, ω) . Put

$$P_G(M, \omega) = \{\varphi \in C^\infty(M) : \omega_\varphi > 0, \text{ and } \varphi \text{ is } G\text{-invariant}\}. \quad (11)$$

By $\omega_\varphi > 0$ we mean that ω_φ is a Kähler metric. If $G = \{1\}$ we simply write $P(M, \omega)$. We say that F_ω is *proper* on $P_G(M, \omega)$ if there is a proper increasing function $\mu : \mathbb{R} \rightarrow \mathbb{R}$, such that the inequality

$$F_\omega(\varphi) \geq \mu(J_\omega(\varphi))$$

holds for any $\varphi \in P_G(M, \omega)$. The importance of this notion is mainly due to the following theorem (see [24, Theorem 1.6] and [25, Chapter 7]).

Theorem 2.1 (Tian) *Let M be a Fano manifold, G a maximal compact subgroup of $\text{Aut}(M)$ and ω a G -invariant Kähler metric in the class $2\pi c_1(M)$. Then M admits a Kähler-Einstein metric if and only if F_ω is proper on $P_G(M, \omega)$. Moreover, in this case F_ω is bounded from below on all $P(M, \omega)$.*

The elements of $P_G(M, \omega)$ parametrise metrics only up to a constant, because ω_φ does not change by adding a constant to φ , and the functional F_ω depends on $\varphi \in P_G(M, \omega)$ only up to a constant. Therefore we can normalise the elements of $P_G(M, \omega)$ one way or another. The following normalisation is useful in this context:

$$Q_G(M, \omega) = \{\varphi \in P_G(M, \omega) : A_\omega(\varphi) = 0\}. \quad (12)$$

For any $\varphi \in P_G(M, \omega)$, $\varphi + A_\omega(\varphi) \in Q_G(M, \omega)$ is the corresponding normalised potential.

The following proposition gives a sufficient condition for the existence of Kähler-Einstein metrics on Fano manifolds.

Proposition 2.1 *Let M be a Fano manifold, ω a Kähler metric in the class $2\pi c_1(M)$ and G a compact group of isometries of (M, ω) . If there are constants $C_1, C_2 > 0$ such that*

$$F_\omega(\varphi) \geq C_1 \sup_M \varphi - C_2 \quad (13)$$

for any $\varphi \in Q_G(M, \omega)$, then M admits a Kähler-Einstein metric.

Proof. One exploits the same estimates used in the proof of Theorem 2.1 (compare [25, Chapter 7].) Indeed, let φ_t , $t \in [0, T)$ be the curve of potentials obtained by applying the continuity method:

$$(\omega + i\partial\bar{\partial}\varphi_t)^n = e^{f-t\varphi_t}\omega^n. \quad (14)$$

Then it is known that for some constants $C_3, C_4 > 0$

$$F_\omega^0(\varphi_t) \leq 0 \quad (15)$$

$$F_\omega(\varphi_t) \leq -A_\omega(\varphi_t) \leq \frac{1-t}{V} \int_M \varphi_t \omega_t^n \quad (16)$$

$$0 \leq -\inf_M \varphi_t \leq C_3 \left(\frac{1}{V} \int_M (-\varphi_t) \omega_t^n + C_4 \right) \quad (17)$$

$$\frac{1}{V} \int_M \varphi_t \omega_t^n \leq C_4 \quad (18)$$

$$F_\omega(\varphi_t) \leq -A_\omega(\varphi_t) \leq C_4(1-t) \leq C_4 \quad (19)$$

(see [25, p. 72]). Since $\varphi_t + A_\omega(\varphi_t) \in Q_G(M, \omega)$, and $F_\omega(\varphi_t)$ does not change by adding a constant to φ_t , an application of (13) yields

$$\begin{aligned} F_\omega(\varphi_t) &= F_\omega(\varphi_t + A_\omega(\varphi_t)) \geq \\ &\geq C_1 \sup_M (\varphi_t + A_\omega(\varphi_t)) - C_2 = C_1 \sup_M \varphi_t + C_1 A_\omega(\varphi_t) - C_2 \end{aligned} \quad (20)$$

Therefore using (19)

$$C_1 \sup_M \varphi_t \leq F_\omega(\varphi_t) - C_1 A_\omega(\varphi_t) + C_2 \leq C_4 + C_2 + C_1 C_4$$

Hence $\sup_M \varphi_t$ is uniformly bounded. But from (15)

$$J_\omega(\varphi_t) \leq \frac{1}{V} \int_M \varphi_t \omega_t^n \leq \sup_M \varphi_t.$$

So $J_\omega(\varphi_t)$ is bounded and this is enough to bound the C^0 norm (see [25, p. 67]). Therefore, by Yau's estimates, one can solve equations (14) up to $t = 1$, and $\omega + i\partial\bar{\partial}\varphi_1$ is the Kähler-Einstein metric. Q.D.E.

Lemma 2.3 *Let M be a Fano manifold, and ω a Kähler metric in the class $2\pi c_1(M)$. Then for any $\beta > 0$ there are constants $C_1, C_2 > 0$ such that for any $\varphi \in Q(M, \omega)$*

$$\log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \omega^n \right] \geq C_1 \sup_M \varphi - C_2. \quad (21)$$

Proof. According to one of the basic results of Tian's theory of the α -invariant (see [22, Prop. 2.1]), there are $\alpha \in (0, 1)$ and $C_3 > 0$, such that for any $\varphi \in P_G(M, \omega)$

$$\frac{1}{V} \int_M e^{-\alpha(\varphi - \sup \varphi)} \omega^n \leq C_3. \quad (22)$$

Let p be such that

$$\frac{p - \alpha}{p - 1} = 1 + \beta. \quad (23)$$

Then

$$p = 1 + \frac{1 - \alpha}{\beta} \quad (24)$$

so $p \in (1, +\infty)$, because $\alpha < 1$. Let $d\mu$ denote the measure $(1/V)\omega^n$ on M . By definition, if $\varphi \in Q_G(M, \omega)$

$$\int e^{f - \varphi} d\mu = 1, \quad (25)$$

so

$$e^{-\sup f} \leq \int e^{-\varphi} d\mu.$$

Since

$$-\varphi = \frac{\alpha}{p}(\sup \varphi - \varphi) - \frac{\alpha}{p} \sup \varphi + \left(1 - \frac{\alpha}{p}\right)(-\varphi)$$

$$e^{-\sup f} \leq e^{-\frac{\alpha}{p} \sup \varphi} \int e^{\frac{\alpha}{p}(\sup \varphi - \varphi)} \cdot e^{\frac{p - \alpha}{p}(-\varphi)} d\mu.$$

Therefore applying Hölder inequality with exponent p yields

$$e^{\frac{\alpha}{p} \sup \varphi - \sup f} \leq \left[\int e^{\alpha(\sup \varphi - \varphi)} d\mu \right]^{1/p} \left[\int e^{\frac{p'}{p}(p - \alpha)(-\varphi)} d\mu \right]^{1/p'}. \quad (26)$$

Using (22) and observing that

$$\frac{p'}{p}(p - \alpha) = 1 + \beta,$$

we get

$$e^{\frac{\alpha}{p} \sup \varphi - \sup f} \leq C_3^{1/p} \left[\frac{1}{V} \int_M e^{-(1 + \beta)\varphi} \omega^n \right]^{1/p'}.$$

Taking logarithms

$$\frac{p'\alpha}{p} \sup \varphi - p' \sup f - \frac{p'}{p} C_3 \leq \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \omega^n \right]$$

that is (21) with

$$C_1 = \frac{p'\alpha}{p} = \frac{\alpha\beta}{1-\alpha} > 0, \quad C_2 = p' \sup f + \frac{C_3}{p-1}.$$

Q.D.E.

Corollary 2.1 *If there are constants $C_1, C_2 > 0$ and $\beta > 0$ such that*

$$F_\omega(\varphi) \geq C_1 \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \omega^n \right] - C_2 \quad (27)$$

for any $\varphi \in Q_G(M, \omega)$, then M admits a Kähler-Einstein metric.

This is an immediate consequence of the previous lemma and Proposition 2.1.

In the proof of the existence theorems below we will need a slight extension of the integral functionals defined above. Let M be a compact complex manifold and γ a continuous hermitian form on M . A closed positive current T of bidegree $(1,1)$ is called a *Kähler current* if for some constant $c > 0$ one has $T \geq c\gamma$ in the sense of currents. The definition does not depend on the choice of γ , since M is compact. If M is a Fano manifold, $G \subset \text{Aut}(M)$ is a compact subgroup, and ω is a G -invariant Kähler form in the class $2\pi c_1(M)$, we put

$$P_G^0(M, \omega) = \{\psi \in C^0(M) : \omega + i\partial\bar{\partial}\psi \text{ is a Kähler current}\}.$$

This means that ψ belongs to $P_G^0(M, \omega)$ if and only if $\omega + i\partial\bar{\partial}\psi \geq c\omega$ in the sense of currents for some $c > 0$.

Lemma 2.4 *The map $\varphi \mapsto (\omega + i\partial\bar{\partial}\varphi)^n$ can be extended to a map*

$$P_G^0(M, \omega) \longrightarrow \{\text{positive Borel measures on } M\}.$$

The extension is continuous with respect to the C^0 -topology on the domain and the weak convergence of measures on the target.

Proof. This follows from basic results on the complex Monge-Ampère operator. Consider a covering $\{U_k\}$ of M with contractible open subsets. On U_k we have $\omega = i\partial\bar{\partial}u_k$ for some smooth strictly plurisubharmonic function u_k . If $\varphi \in P^0$ then $u_k + \varphi$ is plurisubharmonic and continuous on U_k . Although in general currents cannot be multiplied, Bedford and Taylor showed how to define consistently $(i\partial\bar{\partial}(u_k + \varphi))^n$ as a positive measure on U_k . Moreover, it follows from the Chern-Levine-Nirenberg inequality that this measure depends continuously on φ (see e.g. [7, Corollary 2.6]). As $(i\partial\bar{\partial}(u_k + \varphi))^n = (i\partial\bar{\partial}(u_j + \varphi))^n$ on $U_k \cap U_j$, these local measures glue together, and the resulting measure on M , denoted by $(\omega + i\partial\bar{\partial}\varphi)^n$ depends continuously on φ . Q.D.E.

Proposition 2.2 *The functionals I_ω , J_ω , F_ω^0 and F_ω can be extended to $P_G^0(M, \omega)$. The extensions are continuous with respect to the C^0 -topology.*

Proof. It follows from the previous lemma that we can extend continuously I_ω . Using formula (2) we can extend continuously J_ω , and therefore F_ω^0 . $A_\omega(\varphi)$ can be clearly extended continuously to $P_G^0(M, \omega)$. Q.D.E.

In the proof of the next Theorem we will need the following density result.

Proposition 2.3 *Any $\psi \in P_G^0(M, \omega)$ is the C^0 -limit of a sequence $\varphi_n \in P_G(M, \omega)$.*

This is a straightforward application of a result due to Richberg ([20]) that we quote in the version given by Demailly ([6, Lemma 2.15]).

Lemma 2.5 (Richberg) *Let $\psi \in C^0(M)$ be such that $i\partial\bar{\partial}\psi \geq \alpha$ for some continuous $(1,1)$ -form α . Then given any hermitian form γ and any $\varepsilon > 0$, there is a function $\psi' \in C^\infty(M)$ such that $\psi \leq \psi' < \psi + \varepsilon$ and $i\partial\bar{\partial}\psi' \geq \alpha - \varepsilon\gamma$.*

The following two lemmata deal specifically with coverings.

Lemma 2.6 *If $\pi : M \rightarrow N$ is a finite holomorphic map of compact complex manifolds, the direct image via π of a Kähler current on M is a Kähler current on N .*

Proof. Let $R \subset M$ and $B \subset N$ denote ramification and branching locus of π , and d its degree. Let γ_M and γ_N be continuous hermitian forms on M and N respectively. Since $\pi^*\gamma_N$ is continuous and γ_M is positive definite, there is $c_1 > 0$ such that $\gamma_M \geq c_1\pi^*\gamma_N$. If T is a Kähler current on M , by definition $T \geq c_2\gamma_M$ for some $c_2 > 0$, so that $T \geq c\pi^*\gamma_N$ with $c = c_1c_2 > 0$. Given a positive form $\eta \in \Lambda^{n-1,n-1}(N)$ we have

$$\begin{aligned} \langle \pi_*T, \eta \rangle &= \langle T, \pi^*\eta \rangle \geq c \langle \pi^*\gamma_N, \pi^*\eta \rangle = c \int_M \pi^*(\gamma_N \wedge \eta) = \\ &= c \int_{M \setminus R} \pi^*(\gamma_N \wedge \eta) = c \cdot d \int_{N \setminus B} \gamma_N \wedge \eta = c \cdot d \int_N \gamma_N \wedge \eta = c \cdot d \langle \gamma_N, \eta \rangle \end{aligned}$$

so that $T \geq c \cdot d \gamma_N$. This proves the lemma. Q.D.E.

Lemma 2.7 *Let $\pi : M \rightarrow N$ be a degree d covering between n -dimensional Kähler manifolds. Let ω_N be a Kähler metric on N , and $\psi \in P^0(N, \omega_N)$ a continuous potential such that $\pi^*\psi$ be a smooth function on M . Then*

$$F_{\pi^*\omega_N}^0(\pi^*\psi) = F_{\omega_N}^0(\psi). \quad (28)$$

Proof. Put $V_N = \langle N, [\omega_N]^n \rangle$. Then $\langle M, [\pi^*\omega_N]^n \rangle = d \cdot V_N$.

$$\begin{aligned} I_{\omega_N}(s\psi) &= \frac{1}{V_N} \int_N s\psi [\omega_N^n - (\omega_N + s i \partial \bar{\partial} \psi)^n] = \\ &= \frac{1}{dV_N} \int_M s\pi^*\psi [(\pi^*\omega_N)^n - (\pi^*\omega_N + s i \partial \bar{\partial} \pi^*\psi)^n] = \\ &= I_{\pi^*\omega_N}(s\pi^*\psi) \\ J_{\omega_N}(\psi) &= \int_0^1 \frac{I_{\omega_N}(s\psi)}{s} ds = \int_0^1 \frac{I_{\pi^*\omega_N}(s\pi^*\psi)}{s} ds = \\ &= J_{\pi^*\omega_N}(\pi^*\psi) \end{aligned} \quad (29)$$

$$\frac{1}{V_N} \int_N \psi \omega_N^n = \frac{1}{dV_N} \int_M \pi^*\psi (\pi^*\omega_N)^n. \quad (30)$$

Plugging (29) and (30) in the definition of F^0 we get finally

$$F_{\omega_N}^0(\psi) = F_{\pi^*\omega_N}^0(\pi^*\psi). \quad (31)$$

Mark that the functionals $I_{\omega_N}(\psi)$, $J_{\omega_N}(\psi)$ and $F_{\omega_N}^0(\psi)$ are well-defined because ω_N is a Kähler metric and $\psi \in P^0(N, \omega_N)$. On the other hand, $\pi^*\omega_N$ degenerates along the ramification, so it is not a Kähler metric. Nevertheless it is a smooth closed $(1,1)$ -form and $\pi^*\psi$ is a smooth function, so the functionals $I_{\pi^*\omega_N}(\pi^*\psi)$, $J_{\pi^*\omega_N}(\pi^*\psi)$ and $F_{\pi^*\omega_N}^0(\pi^*\psi)$ are well-defined too, thanks to the discussion at p. 4. Q.D.E.

Theorem 2.2 *Let M and N be Fano manifolds, $\pi : M \rightarrow N$ a ramified Galois covering of degree d with structure group G , ω_N a Kähler-Einstein metric on N and $\omega \in 2\pi c_1(M)$ a G -invariant Kähler metric. Denote by $R(\pi)$ be the ramification divisor of π (with multiplicities), and assume that numerically (i.e. in homology)*

$$R(\pi) = -\beta K_M$$

for some $\beta \in \mathbb{Q}$. (Since $R(\pi)$ is effective and $-K_M$ is ample, $\beta > 0$.) Then there is a constant C such that for any $\varphi \in P_G(M, \omega_M)$

$$F_{\omega}^0(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \pi^*\omega_N^n \right] - C. \quad (32)$$

(Here $V = \langle [\omega]^n, [M] \rangle$.)

Proof. The classical Hurwitz formula for the canonical bundle of a ramified covering, $\pi^*K_N = K_M - R(\pi)$, yields that

$$\pi^*[\omega_N] = (1 + \beta)[\omega].$$

Choose a G -invariant $u \in C^\infty(M)$ such that

$$\pi^*\omega_N = (1 + \beta)\omega + i\partial\bar{\partial}u. \quad (33)$$

We claim that any $\varphi \in P_G(M, \omega)$ is of the form

$$\varphi = \frac{u + \pi^*\psi}{1 + \beta}$$

for some $\psi \in P^0(N, \omega_N)$. Indeed $(1 + \beta)\varphi - u$ is G -invariant, so $(1 + \beta)\varphi - u = \pi^*\psi$ for some continuous function ψ , because $N = M/G$ has the quotient topology. Since (as currents)

$$\omega_N + i\partial\bar{\partial}\psi = \frac{1 + \beta}{d} \pi_*(\omega + i\partial\bar{\partial}\varphi),$$

Lemma 2.6 implies that $\omega_N + i\partial\bar{\partial}\psi$ is a Kähler current, i.e. that $\psi \in P^0(N, \omega_N)$. We have shown that to any potential $\varphi \in P_G(M, \omega)$ corresponds a *continuous* potential $\psi \in P^0(N, \omega_N)$ such that

$$\pi^*(\omega_N + i\partial\bar{\partial}\psi) = (1 + \beta)(\omega + i\partial\bar{\partial}\varphi). \quad (34)$$

Since N is Kähler-Einstein by hypothesis, Tian's Theorem 2.1 implies that there is a constant C_3 such that $F_{\omega_N}(\eta) \geq -C_3$ for any $\eta \in P(N, \omega_N)$. By Proposition 2.2 the functional F_{ω_N} can be extended continuously to $P^0(N, \omega_N)$, and by Proposition 2.3 $P(N, \omega_N)$ is dense in $P^0(N, \omega_N)$, so we can conclude that

$$F_{\omega_N}(\psi) \geq -C_3 \quad (35)$$

for ψ as in (34). To finish the proof we need to “lift” this inequality from N to M . From (33) and (9) of Lemma 2.2, applied to the forms $(1 + \beta)\omega$ and $\pi^*\omega_N$, it follows that

$$F_{(1+\beta)\omega}^0((1 + \beta)\varphi) = F_{(1+\beta)\omega}^0(u) + F_{\pi^*\omega_N}^0(\pi^*\psi).$$

Since u does not depend on φ , $F_{(1+\beta)\omega}^0(u)$ is a constant. Next (8) in Lemma 2.2 implies that

$$F_{(1+\beta)\omega}^0((1 + \beta)\varphi) = (1 + \beta)F_\omega^0(\varphi).$$

So

$$F_\omega^0(\varphi) = \frac{1}{1 + \beta} F_{\pi^*\omega_N}^0(\pi^*\psi) - C_4.$$

Using Lemma 2.7 and (35) we get

$$\begin{aligned} F_\omega^0(\varphi) &= \frac{1}{1 + \beta} F_{\omega_N}^0(\psi) - C_4 \geq \\ &\geq \frac{1}{1 + \beta} \log \left[\frac{1}{V_N} \int_N e^{-\psi} \omega_N^n \right] - C_5. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{V_N} \int_N e^{-\psi} \omega_N^n &= \frac{1}{dV_N} \int_M e^{-\pi^*\psi} (\pi^*\omega_N)^n \\ dV_N &= \langle \pi^*[\omega_N^n], [M] \rangle = (1 + \beta)^n V \\ \frac{1}{V_N} \int_N e^{-\psi} \omega_N^n &= \frac{1}{(1 + \beta)^n V} \int e^{-(1+\beta)\varphi} e^u (\pi^*\omega_N)^n \geq \\ &\geq \frac{e^{\inf u}}{(1 + \beta)^n V} \int e^{-(1+\beta)\varphi} (\pi^*\omega_N)^n. \end{aligned}$$

Therefore

$$F_{\omega}^0(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} (\pi^* \omega_N)^n \right] - C_6.$$

Q.D.E.

The first criterion for the existence of Kähler-Einstein metrics is the following

Theorem 2.3 *Let M be an n -dimensional Fano manifold. Assume that ramified coverings $\pi_i : M \rightarrow M_i$ are given for $i = 1, \dots, k$, satisfying the following assumptions:*

1. M_i is a Fano manifold and admits a Kähler-Einstein metric;
2. the coverings are Galois, i.e. $M_i = M/G_i$ for some finite group G_i ,
3. the groups G_i are contained in some compact subgroup $G \subset \text{Aut}(M)$;
4. if $R(\pi_1), \dots, R(\pi_k)$ are the ramification divisors, then

$$\bigcap_{i=1}^k R(\pi_i) = \emptyset;$$

5. the divisors $R(\pi_i)$ are all proportional to the anticanonical divisor of M , i.e. there are some (necessarily positive) rational numbers β_i such that numerically (i.e. in homology)

$$R(\pi_i) = -\beta_i K_M.$$

Then M has a Kähler-Einstein metric.

Proof. Fix a G -invariant Kähler form $\omega \in 2\pi c_1(M)$ and Kähler-Einstein metrics ω_i on M_i . As $G_i \subset G$ we have that

$$P_G(M, \omega) \subset \bigcap_{i=1}^k P_{G_i}(M, \omega). \quad (36)$$

From Theorem 2.2 it follows that for some constants $C_{1i} \in \mathbb{R}$ we have

$$F_{\omega}^0(\varphi) \geq \frac{1}{1+\beta_i} \log \left[\frac{1}{V} \int_M e^{-(1+\beta_i)\varphi} (\pi_i^* \omega_i)^n \right] - C_{1i} \quad (37)$$

for all $\varphi \in P_G(M, \omega)$. Put

$$\begin{aligned} C_1 &= \max C_{1i} & p_i &= 1 + \beta_i & \psi &= e^{-\varphi} \\ \beta &= \min \beta_i & p &= \min p_i = 1 + \beta & d\mu &= \frac{1}{V} \omega^n \end{aligned}$$

and define $\eta_i \in C^\infty(M)$ by

$$\pi_i^* \omega_i^n = \eta_i \omega^n.$$

Clearly $\beta > 0, p > 1$ and $\eta_i \geq 0$. Then (37) becomes

$$F_\omega^0(\varphi) + C_1 \geq \frac{1}{p_i} \log \left[\int_M \psi^{p_i} \eta_i d\mu \right]$$

so

$$F_\omega^0(\varphi) + C_1 \geq \log \|\psi \eta_i^{1/p_i}\|_{p_i}$$

where $\|\cdot\|_s$ denotes the norm of $L^s(M, \mu)$. By construction, $p_i \geq p$, for $i = 1, \dots, k$. If $p_i = p$, then clearly

$$\|\psi \eta_i^{1/p_i}\|_{p_i} \geq C_{2i} \|\psi \eta_i^{1/p}\|_p$$

with $C_{2i} = 1$. If $p_i > p$, then

$$\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q}$$

for some $q > p > 1$. By Hölder inequality

$$\|\psi \eta_i^{1/p}\|_p \leq \|\psi \eta_i^{1/p_i}\|_{p_i} \cdot \|\eta_i^{1/q}\|_q$$

so

$$\|\psi \eta_i^{1/p_i}\|_{p_i} \geq C_{2i} \|\psi \eta_i^{1/p}\|_p$$

with

$$C_{2i} = \frac{1}{\|\eta_i^{1/q}\|_q} > 0.$$

Actually $\|\eta_i^{1/q}\|_q = p_i^{n/q}$ so $C_{2i} = p_i^{-n/q}$. Put $C_2 = \min C_{2i} > 0$. Then

$$\begin{aligned} F_\omega^0(\varphi) + C_1 - \log C_2 &\geq \log \|\psi \eta_i^{1/p}\|_p = \\ &= \frac{1}{p} \log \left[\int_M \psi^p \eta_i d\mu \right] \\ \exp(p F_\omega^0(\varphi) + C_3) &\geq \int_M \psi^p \eta_i d\mu. \end{aligned}$$

Taking the average over i

$$\exp(pF_\omega^0(\varphi) + C_3) \geq \frac{1}{k} \sum_{i=1}^k \int_M \psi^p \eta_i d\mu.$$

Taking the logarithm we get

$$\begin{aligned} F_\omega^0(\varphi) &\geq \frac{1}{p} \log \left[\frac{1}{k} \sum_{i=1}^k \int_M \psi^p \eta_i d\mu \right] - C_4 \\ F_\omega^0(\varphi) &\geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \left(\frac{1}{k} \sum_{i=1}^k \eta_i \right) \omega^n \right] - C_4. \end{aligned} \quad (38)$$

It follows from assumption (4) that for some constant $C_5 > 0$

$$\frac{1}{k} \sum_{i=1}^k \eta_i \geq C_5. \quad (39)$$

Therefore

$$F_\omega^0(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \omega^n \right] - C_6.$$

This holds for any $\varphi \in P_G(M\omega)$. If $\varphi \in Q_G(M, \omega)$, then $F_\omega(\varphi) = F_\omega^0(\varphi)$, so we can apply Corollary 2.1 thus proving the existence of a Kähler-Einstein metric on M . Q.D.E.

The reader will notice that assumption (4) on the ramification divisors is used only to ensure that (39) holds for some constant $C_2 > 0$. This allows to bound

$$\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \omega^n \quad \text{with} \quad \frac{1}{V} \int_M e^{-(1+\beta)\varphi} \sum_{i=1}^k (\pi_i^* \omega_i)^n.$$

If the intersection of the ramification divisors is non-vacuous, the sum of the pull-back measures is degenerate along it. Nevertheless, under some numerical assumptions, it is still possible to bound the integral on the left with the one on the right.

Proposition 2.4 *Let M be an n -dimensional Fano manifold. Assume that ramified coverings $\pi_i : M \rightarrow M_i$ are given for $i = 1, \dots, k$, satisfying the following assumptions:*

1. M_i is a Fano manifold and admits a Kähler-Einstein metric;
2. the coverings are Galois, i.e. $M_i = M/G_i$ for some finite group G_i ;
3. the groups G_i are contained in some compact subgroup $G \subset \text{Aut}(M)$;
4. there are (positive) rational numbers β_i such that numerically

$$R(\pi_i) = -\beta_i K_M.$$

Define $\eta \in C^\infty(M)$ by

$$\frac{1}{k} \sum_{i=1}^k \pi_i^* \omega_i^n = \eta \omega^n, \quad (40)$$

and put

$$c := \sup\{\lambda \geq 0 : \eta^{-\lambda} \in L^1(M, \omega^n)\} \quad (41)$$

and $\beta := \min \beta_i$. If

$$\frac{1}{c} < \beta, \quad (42)$$

then M admits a Kähler-Einstein metric.

Proof of Proposition 2.4. Proceeding as in the proof of Theorem 2.3 one shows that for any $\varphi \in P_G(M, \omega)$

$$F_\omega^0(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \eta \omega^n \right] - C_1. \quad (43)$$

(Compare with equation (38).) It follows from (42) that we can choose a real number s such that

$$1 + \frac{1}{c} < s < 1 + \beta. \quad (44)$$

Put

$$\gamma = \frac{1}{s}(1 + \beta) - 1.$$

It follows that $s > 1$ and $\gamma > 0$. Applying Hölder inequality with exponent s we see that

$$\begin{aligned} \frac{1}{V} \int_M e^{-(1+\gamma)\varphi} \omega^n &= \frac{1}{V} \int_M e^{-(1+\gamma)\varphi} \eta^{1/s} \eta^{-1/s} \omega^n \leq \\ &\leq \left[\frac{1}{V} \int_M e^{-s(1+\gamma)\varphi} \eta \omega^n \right]^{\frac{1}{s}} \cdot \left[\frac{1}{V} \int_M \eta^{-\frac{s'}{s}} \omega^n \right]^{\frac{1}{s'}}. \end{aligned} \quad (45)$$

But (44)

$$\frac{s'}{s} = \frac{1}{s-1} < c$$

so by the definition of c

$$C_2 = \left[\frac{1}{V} \int_M \eta^{-\frac{s'}{s}} \omega^n \right]^{\frac{1}{s'}} < +\infty.$$

On the other hand, $s(1+\gamma) = 1+\beta$, so taking the logarithm on both sides of (45) we get

$$\log \left[\frac{1}{V} \int_M e^{-(1+\gamma)\varphi} \omega^n \right] \leq \frac{1}{s} \log \left[\frac{1}{V} \int_M e^{-(1+\beta)\varphi} \eta \omega^n \right] + \log C_2$$

and applying (43)

$$F_\omega^0(\varphi) \geq \frac{s}{1+\beta} \log \left[\frac{1}{V} \int_M e^{-(1+\gamma)\varphi} \omega^n \right] - C_3.$$

Since $\gamma > 0$ we can still apply Corollary 2.1 to get the existence of a Kähler-Einstein metric. Q.D.E.

It is clear that the last proposition is of some use only if c can be computed or at least bounded from below. This number is an instance of an interesting invariant of a singularity studied - among others - by Demailly and Kollár (see [9] and [15]). Indeed, in the situation of Proposition 2.4, let \mathcal{I} be the ideal sheaf on M that on any coordinate chart U is given by $\mathcal{I} = (f_1, \dots, f_k)$, where $f_1, \dots, f_k \in \mathcal{O}_M(U)$ are local defining equations for the divisors $R(\pi_1), \dots, R(\pi_k)$. The *complex singularity exponent* of \mathcal{I} at a point $x \in U$ is defined as

$$c_x(\mathcal{I}) = \sup\{\lambda \geq 0 : e^{-2\lambda\varphi} \text{ is } L^1 \text{ on a neighbourhood of } x\}, \quad (46)$$

where

$$\varphi = \log(|f_1| + \dots + |f_k|). \quad (47)$$

(See [9, p. 528].) Put

$$c_M(\mathcal{I}) = \inf_{x \in M} c_x(\mathcal{I}). \quad (48)$$

Lemma 2.8 *If c is defined by (41) and \mathcal{I} is the ideal defined above, then $c = c_M(\mathcal{I})$.*

Proof. Let (U, z^1, \dots, z^n) and (V, w^1, \dots, w^n) be coordinate charts on M and M_i respectively, such that $\pi_i(U) \subset V$. Let $w^s = \pi_i^s(z)$ be the local representation of π_i . Then the ramification divisor $R(\pi_i)$ is defined by $f_i = \det(\partial\pi_i^s/\partial z^t)$. On the other hand let

$$\begin{aligned}\omega &= i g_{st} dz^s \wedge d\bar{z}^t \\ \omega_i &= i h_{st} dw^s \wedge d\bar{w}^t\end{aligned}$$

be the local representations of ω and ω_i on M and M_i respectively. It is easy to check that $\pi_i^* \omega_i^n = |f_i|^2 \psi_i \omega^n$, where

$$\psi_i = \frac{|\det(h_{st})|^2}{|\det(g_{st})|^2}.$$

This is a smooth positive function, and by restricting U we can assume that it be bounded and uniformly bounded away from 0. Cover M with a finite collection of open sets U_α such that this holds for all coverings π_1, \dots, π_k . On each such U_α we have

$$\eta = \frac{1}{k} (|f_1|^2 \psi_1 + \dots + |f_k|^2 \psi_k),$$

so for some $C > 0$

$$\frac{1}{C} \eta \leq |f_1|^2 + \dots + |f_k|^2 \leq C \eta.$$

Since $|f_1| + \dots + |f_k| \leq \sqrt{k} \sqrt{|f_1|^2 + \dots + |f_k|^2} \leq \sqrt{k} (|f_1| + \dots + |f_k|)$, the local integrability of $\eta^{-\lambda}$ is equivalent to the local integrability of $e^{-2\lambda\varphi}$ (where φ is defined by (47)). Taking the minimum over α we get the result. Q.D.E.

The complex singularity exponent is in general quite difficult to compute, even for reasonably simple singularities (see [15, §8]). We present below two cases in which the computation is very simple. Although in many other explicit examples it is possible to compute c and to successfully apply Proposition 2.4, a general computation of c seems to be hard, although the singularities of the ramification divisors are relatively mild compared to other kinds of singularities.

We first recall some results on the ramification divisor of a Galois covering.

Lemma 2.9 (Cartan, [5, p. 97]) *Given a finite group G acting holomorphically on a complex manifold M and leaving a point $x \in M$ fixed, there is a biholomorphism between a neighbourhood of x and a neighbourhood of the origin in $T_x M$, that intertwines the action of G and the tangent representation.*

Definition 2.1 *A (pseudo)reflection is a linear map $g \in \mathrm{Gl}(n, \mathbb{C})$ that is diagonalisable and has exactly $n - 1$ eigenvalues equal to 1. A reflection group is a finite subgroup $G \subset \mathrm{Gl}(n, \mathbb{C})$ that is generated by reflections.*

The eigenvalues of a reflection g of finite order (i.e. such that $g^m = 1$) are an m -th root of unity (with multiplicity 1) and 1 (with multiplicity $n - 1$). (When $m = 2$, g is indeed the reflection across its 1-eigenspace.)

Theorem 2.4 (Chevalley-Shephard-Todd) *A finite subgroup $G \subset \mathrm{Gl}(n, \mathbb{C})$ is a reflection group if and only if the affine variety \mathbb{C}^n/G is smooth.*

For the proof of this Theorem we refer to [21, p. 76]. Let now $\pi : M \rightarrow N = M/G$ be a Galois covering and x a point in M . Denote by G_x the stabiliser. Since the action is properly discontinuous, we can find a neighbourhood U_x of x that is G_x -stable and such that $gU_x \cap U_x = \emptyset$ if $g \notin G_x$. By Cartan's lemma we can assume that U_x be isomorphic to some neighbourhood V of the origin in $T_x M$ with the tangent representation. But U_x/G_x is isomorphic to a neighbourhood of $\pi(x)$ in N , and therefore is smooth. Hence, Chevalley-Shephard-Todd's theorem implies that G_x acts on $T_x M$ as a reflection subgroup. Moreover the invariant theory of finite groups provides a nice model for the map $U_x \rightarrow \pi(U_x)$, and in particular ensures that the projection π can be written locally using invariant polynomials: $\pi(z) = (F_1(z), \dots, F_n(z))$. Here F_j is a G_x -invariant polynomial on $T_x M \cong \mathbb{C}^n$ of degree d_i . The polynomial $f = \det(\partial F_i / \partial z^j)$ is a local defining equation for $R(\pi)$. It has degree $(d_1 - 1) + \dots + (d_n - 1)$. On the other hand the (local) degree of the covering π is of course $d_1 \dots d_n$. The inequality $d_1 + \dots + d_n - n \leq d_1 \dots d_n - 1$ implies that in these coordinates the ramification divisor is given by a homogeneous polynomial f whose degree is strictly smaller than the local degree of π , hence a fortiori smaller than the global degree of π . Thus we have proved the following.

Lemma 2.10 *If $\pi : M \rightarrow N$ is a Galois covering between smooth complex manifolds with structure group G , then in appropriate coordinate charts around an arbitrary point the local defining equation of the ramification divisor is a homogeneous polynomial of degree less than $\#G$.*

The description of the ramification divisor can be made more precise (see [21, Exercise 4.3.5 p. 85]). Let H be a hyperplane in \mathbb{C} . The reflections in G_x that fix H form a cyclic group. Denote by $e(H)$ its order, and denote by ℓ_H a linear function on \mathbb{C}^n such that $H = \{\ell_H = 0\}$. Since there are a finite number of reflections there are a finite number of hyperplanes, say H_1, \dots, H_N , that are fixed by some reflection in G_x . Then on U_x the ramification divisor has the following local defining equation:

$$f = \prod_{i=1}^N \ell_{H_i}^{e(H_i)-1} = 0. \quad (49)$$

If the (reduced) ramification is smooth there is only one hyperplane. Since $e(H) \leq \#G_x$, we have proved the following.

Lemma 2.11 *Let $\pi : M \rightarrow N$ be a Galois covering between smooth complex manifolds with structure group G . If the reduced divisor associated to the ramification divisor is smooth at $x \in M$, then there is a holomorphic function ℓ defined on some neighbourhood of x , such that $d\ell(x) \neq 0$, and $R(\pi) = \{\ell^m = 0\}$, with $m \leq \#G - 1$.*

We can now give two simple applications of Proposition 2.4.

Theorem 2.5 *Let M be an n -dimensional Fano manifold, and let $\pi : M \rightarrow N$ be a Galois covering with group G onto a Kähler-Einstein manifold N . Assume that homologically $R(\pi) = -\beta K_M$, and that*

$$d - 1 < \beta \quad (50)$$

where $d = \#G = \deg(\pi)$. Then M has a Kähler-Einstein metric.

Proof. Take $x \in M$. If x does not lie in the support of $R = R(\pi)$ then $\eta^{-\lambda}$ is clearly L_{loc}^1 for any positive λ . If x lies in the support of R , Lemma 2.10 implies that in appropriate coordinates centered at x the divisor R has a local defining equation that is a homogeneous polynomial f of degree m , with $m \leq d - 1$. In particular $\text{ord}_x f = m$. The following general result gives a lower bound for $c_x(f)$ (for the proof see [8, Lemma 8.2, p. 438]).

Lemma 2.12 *Let f be a holomorphic function on an open set $U \subset \mathbb{C}^n$. If $x \in U$, then $c_x(f) \geq 1/\text{ord}_x(f)$.*

From this it follows that $c_x(\mathcal{I}) = c_x(f) \geq 1/m \geq 1/(d-1)$ for any point of M . Hence $c = c_M(\mathcal{I}) \geq 1/(d-1)$, and an application of Proposition 2.4 concludes the proof. Q.D.E.

Theorem 2.6 *Let M be an n -dimensional Fano manifold. Assume that ramified coverings $\pi_i : M \rightarrow M_i$ are given for $i = 1, \dots, k$, satisfying the following assumptions:*

1. M_i is a Fano manifold and admits a Kähler-Einstein metric;
2. the coverings are Galois, i.e. $M_i = M/G_i$;
3. the groups G_i are all contained in some fixed compact subgroup $G \subset \text{Aut}(M)$;
4. if V_i denotes the reduced divisor of M associated to the ramification divisor of π_i , then the V_i 's are smooth hypersurfaces, that intersect transversally in a smooth submanifold V ;
5. there are (positive) rational numbers β_i such that

$$R(\pi_i) = -\beta_i K_M,$$

and they satisfy

$$\frac{1}{d_1 - 1} + \dots + \frac{1}{d_k - 1} > \frac{1}{\beta} \quad (51)$$

where $\beta := \min \beta_i$ and $d_i = \#G_i$.

Then M has a Kähler-Einstein metric.

Proof. In order to apply Proposition 2.4 it is necessary to show that

$$c \geq \frac{1}{d_1 - 1} + \dots + \frac{1}{d_k - 1}. \quad (52)$$

By definition $V = V_1 \cap \dots \cap V_k = V(\mathcal{I})_{\text{red}} = V(\sqrt{\mathcal{I}})$. Let x be a point in M . If $x \notin V$ then $e^{-2\varphi(x)}$ is finite (see (47)), and clearly $c_x(\mathcal{I}) = +\infty$. Let $x \in V$. Using Lemma 2.11 we find a neighbourhood U of x and holomorphic functions ℓ_1, \dots, ℓ_k such that $R(\pi_i) = \{\ell_i^{m_i} = 0\}$. Since the V_i 's cross normally, $d\ell_1, \dots, d\ell_k$ are linearly independent, hence

we can find a coordinate system on a neighbourhood U of x such that $\ell_i = z_i$ for $1 \leq i \leq k$. Since $m_i \leq d_i - 1$, in order to prove (52) it is enough to show that the integral

$$I(\lambda) = \int_U (|z_1^{m_1}| + \dots + |z_k^{m_k}|)^{-2\lambda} \omega^n$$

converges for any positive $\lambda < 1/m_1 + \dots + 1/m_k$, i.e. that $c_x(\mathcal{I}) \geq 1/m_1 + \dots + 1/m_k$. Assuming that the coordinate chart maps U into a polydisk Δ^n (where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$), we get the estimate

$$\begin{aligned} I(\lambda) &\leq C_1 \int_{\Delta^n} \frac{1}{(|z_1|^{m_1} + \dots + |z_k|^{m_k})^{2\lambda}} d\mathcal{L}^n = \\ &= C_2 \int_{\Delta^k} \frac{1}{(|z_1|^{m_1} + \dots + |z_k|^{m_k})^{2\lambda}} d\mathcal{L}^k \end{aligned}$$

\mathcal{L}^n being $2n$ -dimensional Lebesgue measure. Using polar coordinates in each disk Δ with $t_i = |z_i|$, we get

$$I(\lambda) \leq C_3 \int_0^1 dt_1 \dots \int_0^1 dt_k \frac{t_1 \dots t_k}{(t_1^{m_1} + \dots + t_k^{m_k})^{2\lambda}}.$$

With the substitution $t_i = s_i^{1/m_i}$

$$I(\lambda) \leq C_4 \int_0^1 ds_1 \dots \int_0^1 ds_k \frac{s_1^{\frac{2}{m_1}-1} \dots s_k^{\frac{2}{m_k}-1}}{(s_1 + \dots + s_k)^{2\lambda}}.$$

If $\lambda < 1/m_1 + \dots + 1/m_k$, we can choose $\lambda_1, \dots, \lambda_k$ such that $0 < \lambda_i < 1/m_i$ and $\lambda = \lambda_1 + \dots + \lambda_k$. Since $s_1 + \dots + s_k \geq s_i$ we get

$$I(\lambda) \leq C_4 \int_0^1 ds_1 \dots \int_0^1 ds_k \prod_{i=1}^k \left[\frac{s_i^{\frac{2}{m_i}-1}}{(s_1 + \dots + s_k)^{2\lambda_i}} \right] \leq C_4 \prod_{i=1}^k \int_0^1 s_i^{2(\frac{1}{m_i}-\lambda_i)-1} ds_i.$$

And this converges since $\frac{1}{m_i} - \lambda_i > 0$ for every i . Q.D.E.

3 Examples

Consider the hypersurface

$$M = \{x_0^d + \dots + x_{k-1}^d + f(x_k, \dots, x_{n+1}) = 0\} \subset \mathbb{P}^{n+1}$$

where f is any homogeneous polynomial of degree d such that M is smooth. Note that this is equivalent to saying that

$$V = M \cap \{x_0 = \dots = x_{k-1} = 0\} \cong \{f = 0\} \subset \mathbb{P}^{n+1-k}$$

be smooth.

Proposition 3.1 *If $k > n + 2 - d$ then M admits a Kähler-Einstein metric.*

Proof. M admits k Galois \mathbb{Z}_d -coverings $\pi_i : M \rightarrow \mathbb{P}^n$ obtained by deleting the i -th coordinate, $\pi(x_0, \dots, x_{n+1}) = (x_0, \dots, \widehat{x_i}, \dots, x_{n+1})$. $G_i = \mathbb{Z}_d$ acts by multiplication by roots of unity on the i -th coordinate of \mathbb{P}^{n+1} . $R(\pi_i) = \{x_i^{d-1} = 0\} = \mathcal{O}(d-1) = -\beta K_M$ with

$$\beta = \frac{d-1}{n+2-d}.$$

Since the groups G_i commute, they generate a subgroup of $\text{Aut}(M)$ which is isomorphic to $G_0 \times \dots \times G_{k-1}$. Therefore they all lie inside this (finite) compact subgroup of $\text{Aut}(M)$. The ramifications are smooth hyperplane sections, and their intersection is the submanifold V above. Therefore a straightforward application of Theorem 2.6 yields the existence of the Kähler-Einstein metric. Q.D.E.

Proposition 3.2 *Let $M \subset \mathbb{P}^{n+m}$ be a complete intersection of m hypersurfaces of degree d , given by equations of the form*

$$F_j(x_0, \dots, x_{n+m}) = a_0^j x_0^d + \dots + a_{k-1}^j x_{k-1}^d + f_j(x_k, \dots, x_{n+m}) = 0, \\ j = 1, \dots, m.$$

I.e. the equations are diagonal in the first k coordinates. If $n+2-d < k$, then M admits a Kähler-Einstein metric.

Proof. We proceed by induction over m . For $m = 1$ it is the last Proposition. Let $m > 1$, and assume that the result is true for intersections of $m-1$ hypersurfaces. If we delete one of the first k coordinates,

for example x_0 , we get a degree d covering $\pi_0 : M \rightarrow M_0 \subset \mathbb{P}^{n+m-1}$ over a manifold with equations

$$a_0^1 F_j - a_0^j F_1 = b_1^j x_1^d + \dots + b_{k-1}^j x_{k-1}^d + h_j(x_k, \dots, x_{k+m}) = 0,$$

$$\text{where} \quad \begin{cases} b_s^j = a_0^1 a_s^j - a_0^j a_s^1 \\ h_j = a_0^1 f_j - a_0^j f_1 \end{cases}$$

Therefore the base of the covering has equations of the same form, but in smaller number. By induction it has a Kähler-Einstein metric. Moreover we can do the same with any other coordinate x_1, \dots, x_{k-1} , so we get k coverings over Kähler-Einstein manifolds. The ramifications are smooth, as well as their intersection, and

$$\beta = \frac{d-1}{n+m+1-md}.$$

Since $n+1+m(1-d) \leq n+2-d < k$, we see that $\beta > (d-1)/k$, and we can apply Theorem 2.6 to get the Kähler-Einstein metric. Q.D.E.

When $d = 2$, i.e. when we are intersecting quadrics, one needs $k = n+1$, which means that all the quadrics are in diagonal form. If $m = 2$, the following result says that in this way we get *all* the intersection of two quadrics.

Theorem 3.1 *If Q_1, Q_2 are quadrics in \mathbb{P}^{n+2} , such that their intersection $M = Q_1 \cap Q_2$ is smooth and n -dimensional, then there is a system of homogeneous coordinates $(x_0 : \dots : x_{n+2})$ such that*

$$\begin{aligned} Q_1 &= \{x_0^2 + \dots + x_{n+2}^2 = 0\} \\ Q_2 &= \{\lambda_0 x_0^2 + \dots + \lambda_{n+2} x_{n+2}^2 = 0\} \end{aligned} \tag{53}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

For this classical result we refer the reader to the detailed proof given by Miles Reid in his PhD thesis [19, p. 36].

Corollary 3.1 *Any smooth intersection of two quadrics $M = Q_1 \cap Q_2$ in \mathbb{P}^{n+2} has a Kähler-Einstein metric.*

Note that this gives the whole moduli space of such manifolds. In fact a result of Fujita says that these manifolds are characterised by simple numerical invariants (see [13, p. 54, Theorem 3.2.5 (iv)]).

Browsing through the list of Fano 3-folds with $\rho = h^{1,1} = 1$ (see e.g. [13, p. 215]) we see that some of them are already defined as coverings. These are the manifolds that Iskovskikh called *hyperelliptic* because the anticanonical linear system $|-K_M|$ determines a morphism that is a double cover onto its image M' . The branching divisor $B \subset M'$ is smooth, and the pairs (M', B) can be classified. The possibilities are the following ones (see [13, p. 33-34]):

- a) $M' = \mathbb{P}^3$ and B is a sextic surface;
- b) $M' = Q^3$ is the 3-dimensional quadric, and B is cut out by a quartic surface;
- c) $M' \subset \mathbb{P}^6$ is a cone over the Veronese surface, and B is cut out by a cubic hypersurface.

Using Theorem 2.6 we will show that the manifolds in (a) and (b) admit a Kähler-Einstein metric. Actually the same holds for analogous coverings in arbitrary dimension. Whether (c) can be dealt with these methods is not clear.

Theorem 3.2 *Let M be an n -dimensional Fano manifold that admits a double covering π over \mathbb{P}^n with branching divisor a smooth hypersurface of degree $2d$, with $\frac{n+1}{2} < d \leq n$. Then M admits a Kähler-Einstein metric.*

Proof. That these coverings are smooth depends on the fact that the branching divisor is smooth, see [3, p. 42]. Recall that given for a double cover $\pi : M \rightarrow N$, the ramification and branching divisors are related by

$$R = \frac{1}{2}\pi^*B. \quad (54)$$

Since $B = \mathcal{O}(2d)$, $R = \pi^*\mathcal{O}(d)$ and it follows from Hurwitz formula that $-K_M = \pi^*\mathcal{O}(n+1-d)$. Therefore $R = -\beta K_M$ with

$$\beta = \frac{d}{n+1-d}. \quad (55)$$

As the pull-back of an ample line bundle by a finite map is ample, these manifolds are Fano for $1 \leq d \leq n$. In order to apply Theorem 2.5 we only need to check that (50) holds, with $d = 2$, i.e. $1 < \beta$. And this holds if and only if $d > (n+1)/2$. Q.D.E.

Theorem 3.3 *Let M be an n -dimensional Fano manifold that is a double cover of the quadric $Q_n \subset \mathbb{P}^n$ ramified along a smooth divisor cut out by a hypersurface of degree $2d$, with $\frac{n}{2} < d < n$. Then M admits a Kähler-Einstein metric.*

Proof. Denote by $\pi : M \rightarrow Q_n$ the covering and by $i : Q_n \hookrightarrow \mathbb{P}^{n+1}$ the inclusion. Put $\varphi = i\pi$. Then $B = i^*\mathcal{O}(2d)$, and $R = \frac{1}{2}\pi^*B = \varphi^*\mathcal{O}(d)$, while $-K_M = \varphi^*\mathcal{O}(n) - \varphi^*\mathcal{O}(d) = \varphi^*\mathcal{O}(n - d)$. So for $d < n$ M is Fano and

$$\beta = \frac{d}{n - d}.$$

When $2d > n$, $\beta > 1$, and Theorem 2.5 yields the result. Q.D.E.

In case $n = 3$, d has to be equal to 2, i.e. B is cut out by a quartic, and the branching divisor is an octic hypersurface contained in Q_3 . When $d = 1$ (and n arbitrary), it is not possible to apply Theorem 2.6, but in this case the manifold M is simply the intersection of two quadrics.

References

- [1] Claudio Arezzo and Andrea Loi. Moment maps, scalar curvature and quantization of Kähler manifolds. *Comm. Math. Phys.*, 246(3):543–559, 2004.
- [2] D. Avritzer and H. Lange. Pencils of quadrics, binary forms and hyperelliptic curves. *Comm. Algebra*, 28(12):5541–5561, 2000. Special issue in honor of Robin Hartshorne.
- [3] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.
- [4] Charles P. Boyer, Krzysztof Galicki, and János Kollár. Einstein metrics on spheres. *arXiv:math.DG/0309408*, 2003. Preprint, to appear in *Annals of Mathematics*.
- [5] H. Cartan. Quotient d’un espace analytique par un groupe d’automorphismes. In *Algebraic geometry and topology*, pages 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz,.

- [6] J.-P. Demailly. Regularization of closed positive currents and intersection theory. *J. Algebraic Geom.*, 1(3):361–409, 1992.
- [7] J.-P. Demailly. Monge-Ampère operators, Lelong numbers and intersection theory. In *Complex analysis and geometry*, Univ. Ser. Math., pages 115–193. Plenum, New York, 1993.
- [8] J.-P. Demailly. *Complex Analytic and Differential Geometry*. 1997. available at the internet page of the author, www-fourier.ujf-grenoble.fr/~demailly.
- [9] Jean-Pierre Demailly and János Kollár. Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. École Norm. Sup. (4)*, 34(4):525–556, 2001.
- [10] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In *Northern California Symplectic Geometry Seminar*, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.
- [11] S. K. Donaldson. Scalar curvature and projective embeddings. I. *J. Differential Geom.*, 59(3):479–522, 2001.
- [12] S. K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differential Geom.*, 62(2):289–349, 2002.
- [13] V. A. Iskovskikh and Yu. G. Prokhorov. Fano varieties. In *Algebraic geometry, V*, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999.
- [14] J. M. Johnson and J. Kollár. Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces. *Ann. Inst. Fourier (Grenoble)*, 51(1):69–79, 2001.
- [15] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.
- [16] T. Mabuchi and S. Mukai. Stability and Einstein-Kähler metric of a quartic del Pezzo surface. In *Einstein metrics and Yang-Mills connections (Sanda, 1990)*, volume 145 of *Lecture Notes in Pure and Appl. Math.*, pages 133–160. Dekker, New York, 1993.
- [17] A. M. Nadel. Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. *Ann. of Math. (2)*, 132(3):549–596, 1990.
- [18] Sean T. Paul and Gang Tian. Analysis of geometric stability. *Int. Math. Res. Not.*, (48):2555–2591, 2004.

- [19] Miles Reid. The complete intersection of two or more quadrics. *PhD thesis, Trinity College, Cambridge*, 1972. available at the internet page of the author, www.maths.warwick.ac.uk/.
- [20] R. Richberg. Stetige streng pseudokonvexe Funktionen. *Math. Ann.*, 175:257–286, 1968.
- [21] T. A. Springer. *Invariant theory*. Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 585.
- [22] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. *Invent. Math.*, 89(2):225–246, 1987.
- [23] G. Tian. On Calabi’s conjecture for complex surfaces with positive first Chern class. *Invent. Math.*, 101(1):101–172, 1990.
- [24] G. Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130(1):1–37, 1997.
- [25] G. Tian. *Canonical metrics in Kähler geometry*. Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld.
- [26] G. Tian and S.-T. Yau. Kähler-Einstein metrics on complex surfaces with $C_1 > 0$. *Comm. Math. Phys.*, 112(1):175–203, 1987.
- [27] Xu-Jia Wang and Xiaohua Zhu. Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.